A Colorful Determinantal Identity, a Conjecture of Rota, and Latin Squares

Shmuel Onn
Department of Operations Research
Technion - Israel Institute of Technology
32000 Haifa, Israel
onn@ie.technion.ac.il

1 Rota's Colorful Conjecture and the Latin Square Conjecture

The following conjecture in combinatorial linear algebra is due to Gian-Carlo Rota.

Rota's Colorful Conjecture. Let ${}^1W, \ldots, {}^nW$ be bases of an n-dimensional vector space. Then their multiset union can be repartitioned into bases ${}^1U, \ldots, {}^nU$ such that $|\,{}^iU \cap {}^jW| = 1$ for all i, j.

Regarding the vectors in each ${}^iW$ as *colored* in color i, the newly sought bases are *colorful*, namely contain one vector of each color. So Rota's Colorful Conjecture is that any n colored bases of an n-dimensional vector space can be repartitioned into n colorful bases.

A *Latin square* of order n is an n by n matrix $L = (L_i^j)$ in which each row and each column is a permutation of $\{1, \ldots, n\}$. More precisely, there are permutations $\sigma_1, \ldots, \sigma_n$ and $\pi_1, \ldots, \pi_n$ such that $L_i^j = \sigma_i(j) = \pi_j(i)$ for all i, j. The *sign* of the Latin square is defined as the product of all signs of its row and column permutations

$$\mathrm{sgn}(L) = \prod_{i=1}^{n} \mathrm{sgn}(\sigma_i) \cdot \mathrm{sgn}(\pi_i).$$

A Latin square is *even* if its sign is positive, and *odd* otherwise. Let $l(n)$ be the number of even Latin squares of order n minus the number of odd ones. It is easy to see that $l(n) = 0$ for all odd n. The following conjecture is due to Noga Alon and Michael Tarsi (cf. [1]).

Latin Square Conjecture. The number of even Latin squares of order n minus the number of odd ones satisfies $l(n) \neq 0$ for all even n.

This conjecture has been recently proved by Drisko [4] for all $n = p+1$ where p is a prime. The exact values of $l(n)$ are known only for $n \leq 8$ [7]. It is plausible that $l(n)$ is in fact always nonnegative.

In this note we establish an identity, the *Colorful Determinantal Identity*, which links the two conjectures. It shows that for any n, the Latin Square Conjecture implies Rota's Colorful Conjecture. In particular, Rota's Colorful Conjecture is true for any $n = p+1$ where p is a prime. To compactly express the identity, let S_n be the symmetric group of permutations on $\{1, \ldots, n\}$ and denote by $\mathcal{S}^n$ the collection of n-tuples $\rho = (\rho_1, \ldots, \rho_n)$ of permutations. For $\rho \in \mathcal{S}^n$ let $\operatorname{sgn}(\rho) = \prod_{i=1}^n \operatorname{sgn}(\rho_i)$. For a matrix W let W^j be its j-th column vector. The proof of the following theorem is given in Section 2.

Theorem 1 (Colorful Determinantal Identity). Let ${}^1W, \ldots, {}^nW$ be n square matrices of order n over an arbitrary field. Then

$$\sum_{\rho \in \mathcal{S}^n} \operatorname{sgn}(\rho) \prod_{i=1}^n \det\left({}^1W^{\rho_1(i)}, \ldots, {}^nW^{\rho_n(i)} \right) = l(n) \cdot \prod_{j=1}^n \det({}^jW).$$

Note that for each $\rho \in \mathcal{S}^n$, the tuples $\left({}^1W^{\rho_1(i)}, \ldots, {}^nW^{\rho_n(i)}\right)$, $i = 1, \ldots, n,$ which appear in the left hand side of the identity, give a colorful repartition of the multiset of column vectors of the ${}^jW$.

Corollary 1. For any even n the Latin Square Conjecture implies Rota' Colorful Conjecture over any field of characteristic which does not divide $l(n)$ (in particular characteristic zero).

Proof of Corollary 1 from Theorem 1. Suppose that $l(n) \neq 0$ and let ${}^1W, \ldots, {}^nW$ be given bases, i.e. nonsingular matrices of order n over a field satisfying the hypothesis. Then the right hand side of the Colorful Determinantal Identity is nonzero. Therefore, on the left hand side, there must exist a nonzero summand

$$\prod_{i=1}^n \det\left({}^1W^{\rho_1(i)}, \ldots, {}^nW^{\rho_n(i)} \right), \quad \text{and so the sets} \quad {}^iU = \left\{ {}^1W^{\rho_1(i)}, \ldots, {}^nW^{\rho_n(i)} \right\}, \quad i = 1, \ldots, n,$$

give the desired repartition into colorful bases. ∎

Corollary 1 had been independently obtained by Huang and Rota, but their derivation is quite complex and indirect and involves an intermediate conjecture of Rota on a certain straightening coefficient in the so-called *supersymmetric bracket algebra* [6].

Rota's Colorful Conjecture has a natural generalization to *matroids* [6] which had been verified only for $n = 3$ [3]. Another generalization of Rota's conjecture is a conjecture of Jeff Kahn (cf. [6]) that concerns n^2 bases, for which we have derived a determinantal identity which is reminiscent of Theorem 1. The special case of Kahn's conjecture in which the vectors are in *general position* is known as the *Dinitz problem* and was recently settled affirmatively by Galvin [5]. We refer the reader to [2] for some related algorithmic problems and a discussion of their computational complexity.

2 The Colorful Determinantal Identity

Theorem 1 (Colorful Determinantal Identity). Let ${}^1W, \ldots, {}^nW$ be n square matrices of order n over an arbitrary field. Then

$$\sum_{\rho \in \mathcal{S}^n} \operatorname{sgn}(\rho) \prod_{i=1}^{n} \det\left({}^1W^{\rho_1(i)}, \ldots, {}^nW^{\rho_n(i)} \right) = l(n) \cdot \prod_{j=1}^{n} \det({}^jW).$$

Proof. For a matrix W let W^j be its j-th column vector as before, and let W_i denote its i-th row vector and W_i^j its (i,j)-th entry. Given n square matrices ${}^1W, \ldots, {}^nW$ of size n, define the following polynomial in their entries:

$$\Delta = \sum_{\sigma, \rho \in \mathcal{S}^n} \operatorname{sgn}(\sigma)\operatorname{sgn}(\rho) \prod_{i,j=1}^{n} {}^jW_{\sigma_i(j)}^{\rho_j(i)}.$$

For each ρ and σ in $\mathcal{S}^n$ define

$$\Delta^\rho = \sum_{\sigma \in \mathcal{S}^n} \operatorname{sgn}(\sigma) \prod_{i,j=1}^{n} {}^jW_{\sigma_i(j)}^{\rho_j(i)} = \prod_{i=1}^{n} \sum_{\sigma_i \in S_n} \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n} {}^jW_{\sigma_i(j)}^{\rho_j(i)} = \prod_{i=1}^{n} \det\left({}^1W^{\rho_1(i)}, \ldots, {}^nW^{\rho_n(i)} \right),$$

$$\Delta_\sigma = \sum_{\rho \in \mathcal{S}^n} \operatorname{sgn}(\rho) \prod_{i,j=1}^{n} {}^jW_{\sigma_i(j)}^{\rho_j(i)} = \prod_{j=1}^{n} \sum_{\rho_j \in S_n} \operatorname{sgn}(\rho_j) \prod_{i=1}^{n} {}^jW_{\sigma_i(j)}^{\rho_j(i)} = \prod_{j=1}^{n} \det\left({}^jW_{\sigma_1(j)}, \ldots, {}^jW_{\sigma_n(j)} \right).$$

Now Δ_σ is nonzero only for $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathcal{S}^n$ for which there exists an element $\pi = (\pi_1, \ldots, \pi_n)$ in $\mathcal{S}^n$ such that $\sigma_i(j) = \pi_j(i)$ for all i, j, in which case

$$\Delta_\sigma = \prod_{j=1}^{n} \det\left({}^jW_{\pi_j(1)}, \ldots, {}^jW_{\pi_j(n)} \right) = \operatorname{sgn}(\pi) \prod_{j=1}^{n} \det\left({}^jW_1, \ldots, {}^jW_n \right) = \operatorname{sgn}(\pi) \prod_{j=1}^{n} \det({}^jW).$$

Each $\sigma \in \mathcal{S}^n$ for which such $\pi \in \mathcal{S}^n$ exists gives a Latin square L via $L_i^j = \sigma_i(j) = \pi_j(i)$, whose sign is given by $\operatorname{sgn}(L) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)$. Denote by $\mathcal{L}$ the set of all Latin squares of order n. Then

$$\sum_{\rho \in \mathcal{S}^n} \operatorname{sgn}(\rho)\Delta^\rho = \Delta = \sum_{\sigma \in \mathcal{S}^n} \operatorname{sgn}(\sigma)\Delta_\sigma = \sum_{L \in \mathcal{L}} \operatorname{sgn}(L) \prod_{j=1}^{n} \det({}^jW) = l(n) \prod_{j=1}^{n} \det({}^jW),$$

which is precisely the Colorful Determinantal Identity. □

Acknowledgment

The author thanks Noga Alon and Ron Holzman for some related discussions, and the Alexander von Humboldt Stiftung and the Fund for the Promotion of Research at the Technion for their suuport.

References

[1] N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica, 12:125–134, 1992.

[2] I. Bárány and S. Onn, Colorful linear programming, Lecture Notes in Computer Science, 1084:1–15, 1996, Springer-Verlag.

[3] W. Chan, An exchange property of matroid, Discr. Math., 146:299–302, 1995.

[4] A.A. Drisko, On the number of even and odd Latin squares of order $p+1$, Adv. Math., to appear.

[5] F. Galvin, The list chromatic index of a bipartite multigraph, J. Combin. Theory Ser. B, 63:153–158, 1995.

[6] R. Huang and G.C. Rota, On the relations of various conjectures on Latin squares and straightening coefficients, Discr. Math., 128:225–236, 1994.

[7] J.C.M. Janssen, On even and odd Latin squares, J. Combin. Theory Ser. A, 69:173–181, 1995.