SUPPLEMENTARY MATERIAL FOR

"WEAK MONOTONICITY CHARACTERIZES DETERMINISTIC DOMINANT-STRATEGY IMPLEMENTATION"

by

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In our main paper, we define a weak-monotonicity (W-Mon) condition that is necessary and sufficient for dominant-strategy implementation in a variety of domains. This supplementary material complements the discussion there by providing additional examples and proofs. The notation used here is defined in the paper.

Example S1 demonstrates that the assumption that the range of the social choice function is finite is crucial. In particular, it implies that W-Mon is not a sufficient condition for social choice rules which map reported types into the set of probability distributions over outcomes.

EXAMPLE S1: (W-MON IS NOT SUFFICIENT FOR RANDOM SOCIAL CHOICE FUNCTIONS)[1] There are two identical units and one buyer whose (marginal) valuation vector for the two units is $v = (v_1, v_2) \in [0,1]^2$. (The buyer's utility for 0 units is normalized to zero.) Let $G = (g_1, g_2)$ be a random social choice function where $g_k(\cdot)$ is the probability of allocating at least k units, $k = 1$ or 2, to the buyer as a function of his type.[2] G is W-Mon if

$$[G(v') - G(v)] \cdot (v' - v) \geq 0, \qquad \forall v,\, v'.$$

Define $G(v) = \frac{1}{3}Av$, where A is the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. We have

$$[G(v') - G(v)] \cdot (v' - v) \;=\; 1/3[A(v' - v)] \cdot (v' - v) \;=\; 1/3(v' - v)^T A^T (v' - v) \;\geq\; 0,$$

[1] We are grateful to an anonymous referee for this example.

[2] Note that we use a slightly different representation of a random social function than the one in (Section 5 of) the paper. This representation is convenient for domains with complete orders.

where the inequality follows as A is positive semi-definite. Thus, G is W-Mon. But G cannot be a subgradient of a convex function as the matrix of second partials of this convex function would then be A, which is not possible as A is not symmetric. $\triangle$

Next, we elaborate on the discussion in the paper on the relationship to the work of Roberts (1979) and his PAD condition. While W-Mon implies PAD (Lemma S1) the opposite is not true, even in a single-agent setting (Example S2).

LEMMA S1: *Weak monotonicity implies PAD.*

PROOF: Fix any V, $V' \in D$. Suppose that $f(v) = a^*$ and that the hypothesis of the PAD condition in (13) of the paper is satisfied. That is, $V'(a^*) - V(a^*) > V'(a) - V(a)$ for all $a \in A \backslash \{a^*\}$. Let $V^0 = V$, $V^i = (V_1', V_2', \ldots, V_i', V_{i+1}, \ldots, V_n)$, $i = 1, 2, ..., n$. By repeated application of W-Mon, $f(V^0) = a^*$ implies $f(V^1) = a^*$ which in turn $f(V^2) = a^*$ and so on. Thus, $f(V') = f(V^n) = a^*$. *Q.E.D.*

EXAMPLE S2: (PAD DOES NOT IMPLY W-MON EVEN IN A SINGLE-AGENT MODEL) There are three identical units of the object and one buyer. The marginal utility for the kth unit is v_k, $k = 1, 2, 3$. The mechanism $f(v)$ is defined as follows:

$$f(v_1, v_2, v_3) = \begin{cases} 0, & \text{iff } v_1 = 0 \text{ and } v_3 = 0, \\ 3, & \text{otherwise.} \end{cases} \tag{1}$$

Thus, the buyer either gets nothing or he gets all the three units. Further, $0 \leq v_k \leq 1$, $k = 1, 2, 3$. That is, bounded-domain assumption A is satisfied.

Take any v such that $f(v) = 3$. Let v' be such that v' and v satisfy the hypothesis of the PAD condition (see (13) in the paper). This hypothesis (with $a = 2$ units) implies that $v_3' > v_3 > 0$. Hence $f(v_3') = 3$ by (1). This satisfies the restriction imposed by PAD.

Next, take any v such that $f(v) = 0$. Thus $v_1 = 0$. For $a = 1$ unit in the hypothesis in the PAD inequality, it is impossible that there exists v' such that $v_1' < v_1 = 0$. Thus, no v' satisfies this hypothesis and PAD imposes no restriction whenever $f(v) = 0$. Hence f satisfies PAD.

To see that f is not W-Mon, note that $f(0,1,0) = 0$, $f(0.01,0,0) = 3$. This violates W-Mon as $0.01 + 0 + 0 \not\geq 0 + 1 + 0$. Theorem 2 implies, and we also verify directly, that f is not truth-telling.

Suppose, to the contrary, that there exist prices p_0 and p_3 that induce truth-telling. Clearly, $p_3 \geq p_0$. Since $f(0,0,0) = 0$ and $f(\epsilon,0,0) = 3$ for any $\epsilon > 0$, it must be that $p_3 - p_0 \leq \epsilon$, $\forall \epsilon > 0$. Thus, $p_3 \leq p_0$ and hence $p_3 = p_0$. But then all types $v = (0, v_2, 0)$, $v_2 > 0$, would want to deviate and misreport their type so as to get 3 instead of 0 units. Contradiction. △

Example 1 in the paper shows that W-Mon is not sufficient when the domain of types is finite. One might ask whether a strengthening of W-Mon characterizes incentive compatibility on finite domains. We give below two natural candidates for a stronger condition and an example that shows that neither condition is necessary.

A social choice function f satisfies *strong PAD* if for any V, V', if $f(V) = a$, and $V_i'(a) - V_i(a) \geq V_i'(b) - V_i(b)$, $\forall b \in A$, $i = 1, 2, ..., n$ then $f(V') = a$.

A social choice function f satisfies *generalized W-Mon* if for any V, V', if $f(V) = a$, $f(V') = b$ then there exists an agent i such that $V_i'(b) - V_i(b) \geq V_i'(a) - V_i(a)$.

Example S3: (Strong PAD, generalized W-Mon are not necessary). There are two agents, 1 and 2, and four alternatives $A = \{YY, YN, NY, NN\}$. For any $a \in A$, denote $a = a_1 a_2$ where $a_i = Y$ or N. Agent i's preferences are determined by a_i:

$$V_i(a_1 a_2) \quad = \quad \begin{cases} V_i, & \text{if } a_i = Y \\ 0, & \text{if } a_i = N. \end{cases}$$

Define $f(V) = a_1 a_2$ where $a_i = Y$ if and only if $V_i > 2V_j - 10$ and payment function $p_i(Y, V_j) = 2V_j - 10$, $p_i(N, V_j) = 0$. It is easy to check that (f, p) is dominant-strategy incentive compatible.

Now suppose that for each agent types $V_i^H = 11$ and $V_i^L = 9$ are in the domain. Then $f(V^L) = YY$ but $f(V^H) = HH$. Thus f violates strong PAD and generalized W-Mon. △

With the next example we demonstrate that W-Mon can be used as a tool to check dominant-strategy implementability of many classical social choice rules. In particular, we show that the Rawlsian social choice rule does not satisfy W-Mon. Hence, we have a simple way of demonstrating that it cannot be implemented in dominant strategies.

Example S4: (The Rawlsian social choice function is not W-Mon). There are two agents, 1 and 2, and two heterogenous objects, a and b. The agents have assignment model preferences over the objects. The utilities V_1, V_1' for agent 1 and V_2 for agent 2 are in the domain, with:

$$\begin{aligned} V_1(a) &= 4, & V_1(b) &= 10, \\ V_1'(a) &= 0.5, & V_1'(b) &= 2, \\ V_2(a) &= 1, & V_2(b) &= 2. \end{aligned}$$

At (V_1, V_2) Rawls' max-min rule allocates a to 1 and b to 2 whereas at (V_1', V_2) it allocates b to 1 and a to 2, thus violating W-Mon. Hence, it is not dominant-strategy incentive compatible. △

References

Bikhchandani, S., S. Chatterji, R. Lavi, A. Mu'alem, N. Nisan, and A. Sen (2006): "Weak Monotonicity Characterizes Deterministic Dominant-strategy Implementation," working paper.