Two Simplified Proofs for Roberts' Theorem

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Abstract

Roberts (1979) showed that every social choice function that is ex-post implementable in private value settings must be weighted VCG, i.e. it maximizes the weighted social welfare. This paper provides two simplified proofs for this. The first proof uses the same underlying key-point, but significantly simplifies the technical construction around it, thus helps to shed light on it. The second proof builds on monotonicity conditions identified by Rochet [11] and Bikhchandani et. al. [2]. This proof is for a weaker statement that assumes an additional condition of "player decisiveness".

1 Introduction

Social choice functions represent a way to aggregate individuals' preferences into an integrated social preference. When viewing each individual as a "game-theoretic entity", acting selfishly to maximize its own utility, one needs to provide incentives for the individuals to actually reveal their true preferences. This is commonly done by assuming quasi-linear private value utilities, and introducing a carefully designed payment scheme that induces truthful behavior (in equilibrium). With such a payment scheme, we *implement* the social choice function. The most convincing equilibrium concept is probably that of dominant strategies: no matter what the other players do, player i will maximize his own utility by revealing his true preferences. In private value settings, dominant strategies implementation is equivalent to ex-post implementation, making its importance even more notable. It is therefore important to understand what social choice functions are ex-post implementable (or equivalently dominant-strategy implementable).

To answer this question, one first has to fix a domain of preferences. On one extreme, when the domain is single-dimensional, implementability essentially reduces to a simple monotonicity condition (Myerson [9]) which allows for many types of social choice functions to be implemented. The difficulty comes from switching to a multi-dimensional domain. In such domains, the monotonicity conditions are much more complex, and almost no examples of ex-post implementable functions exist. The only general implementability result is the classic Vickrey-Clarke-Groves (VCG) mechanisms that maximize the social welfare. Other common and natural social goals, like min-max fairness criteria, are not known to be implementable, and, in general, no characterization of the class of ex-post implementable social choice rules is at hand. To further stress the importance of this question, let us parallel it to the case of non-quasi-linear utilities. In that case, Arrow's theorem

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gives the characterization for unrestricted domains; on the other extreme single-peaked domains enable many positive results; and the status for many multi-dimensional domains, like saturated domains [6], is quite understood.

Unfortunately, no such understanding exists for the central model of quasi-linear utilities with private values. The only exception is the work of Roberts [10], that gave the desired characterization for an *unrestricted* domain: the only implementable social choice functions are weighted welfare maximizers! But for all the intermediate range of domains, that are not single-dimensional nor unrestricted, almost no possibilities or impossibilities are known. For example, in the context of auctions, Holzman and Monderer [4] pointed out the potential importance of Roberts' theorem to the understanding of the uniqueness of truth-telling mechanisms in a private value setting.

In light of this, it seems extremely important to fully understand the techniques and intuitions behind Roberts' original proof. To this end, we provide two alternative proofs: in the first, the key point of Roberts' proof (the use of the separation lemma) remains, but the construction process is significantly simplified, thus making the essence visible. The second theorem is a weaker version, with an additional requirement of "player decisiveness" [8]. Our proof relies on the cycle monotonicity characterization of Rochet [11], and shows how to strengthen this characterization, for the case of unrestricted domains, to yield as a result weighted welfare maximization.

1.1 Recent related work

To emphasize the importance of truly understanding the proofs and structures, let us describe the recent efforts that were done in this direction.

The first branch of investigations consider the characterization of truthfulness in terms of monotonicity. Roberts [10] defines a monotonicity condition called PAD, that fully characterizes truthfulness for unrestricted domains. Bikhchandani et. el. [2] identify a weak monotonicity condition that fully characterizes truthfulness for *many restricted domains*. This is further discussed and analyzed also by Lavi, Mu'alem, and Nisan [7], by Gui, Muller, and Vohra [3] and by Saks and Yu [13]. Rochet [11], and independently Rozenshtrom [12], define a "cycle-monotonicity" condition that fully characterizes ex-post implementability for every domain.

Another direction is to investigate these questions for the model of inter-dependent valuations. This is studied by e.g. Jehiel, Meyer-ter-Vehn and Moldovanu [5]. They use Roberts' theorem to characterize ex-post implementable interdependent valuations that are "semi-separable".

The most intriguing question of all is arguably to classify the domains in which weighted VCG's are the only implementable social choice functions. While as mentioned above this question is well-studied for non-quasi-linear settings, and also for quasi-linearity with inter-dependent valuations, the important case of private values is almost unexplored, with few exceptions. Lavi, Mu'alem and Nisan [7] generalize Roberts' characterization for a family of "auction-like" restricted domains, including Combinatorial Auctions and Multi Unit Auctions. They show that, under several additional requirements (of which the crucial one resembles Arrow's IIA condition), every implementable social choice function must be "almost" weighted-VCG. On the other hand, many positive results for implementable social choice functions that are not weighted VCG have been recently introduced by the Computer Science community. Most of these are for single-dimensional domains, but one interesting example of dominant-strategy non-VCG mechanism for a multi dimensional auction domain is given by Bartal, Gonen, and Nisan [1].

1.2 The Formal Setting

Before stating Roberts' characterization, let us first describe the setting. A social designer is required to choose one alternative among a finite set $A = \{x, y, z, ...\}$ of social alternatives. There are n players, each has a private type $v_i \in V_i \subseteq \Re^{|A|}$, where $v_i(x)$ is interpreted as i's resulting value if alternative x were to be chosen, and V_i is the space of all possible types of player i. We denote by $V = V_1 \times \ldots \times V_n$ the type space of all players. The domain V is called *unrestricted* if $V_i = \Re^{|A|}$ for all i, i.e. every $|A|$-tuple of real numbers can represent a valid private type. Let $f : V \to A$ be a social choice function, i.e. f represents the goals of the social designer. Assume w.l.o.g that f is onto A.

In order to motivate the players to reveal their true types, the social designer is allowed to charge payments ($p_i : V \to \Re$) from the players. We assume that players are quasi-linear and rational in the sense of maximizing their total utility: $u_i = v_i(f(v)) - p_i(v)$. We say that f is *truthfully implementable* (in dominant strategies) if there exist payment functions that induce truthfulness as a dominant strategy, for every player. I.e. player i will maximize his utility by declaring his true type v_i, rather than declaring some false type v_i', no matter what the other players declare. Formally, if for every player i, every $v_{-i} \in V_{-i}$, and every $v_i, v_i' \in V_i$: $v_i(f(v)) - p_i(v) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i})$.

Our motivation, therefore, is to understand what social choice functions can or cannot be implemented. It is well known that all (weighted) welfare-maximizers are truthfully implementable, by using Vickrey-Clarke-Groves payments. This is true for *any* domain of players' types and every set of alternatives. For unrestricted domains, Roberts proved the opposite direction:

Theorem 1 (Roberts [10]) *Suppose $|A| \geq 3$ and V is unrestricted domain. Then, for every implementable social choice function f, there exist non-negative weights $k_1, \ldots, k_n$, not all of them equal to zero, and constants $\{C_x\}_{x \in A}$ such that, for all $v \in V$,*

$$f(v) \in argmax_{x \in A} \ \{ \ \Sigma_{i=1}^{n} k_i v_i(x) + C_x \ \}.$$

2 Monotonicity Conditions

Both proofs start by eliminating the need to rely on price specifications, by replacing prices with certain monotonicity conditions, that are necessary to the existence of appropriate prices. Our two proofs start from two different monotonicity conditions, but the origin of both conditions may be viewed as the recently defined property of "weak monotonicity" (W-MON), formalized by Bikhchandani et. al. [2]. We give here a short exposition on these various conditions, partly accompanied with proofs for completeness.

Definition 1 (Bikhchandani et. al. [2]) *A social choice function $f : V \to A$ satisfies* **W-MON** *if $f(v) = x$ and $f(v_i', \ v_{-i}) = y$ implies that:*

$$v_i'(y) - v_i(y) \geq v_i'(x) - v_i(x)$$

for every $v_{-i} \in V_{-i}$ and $v_i, \ v_i' \in V_i$.

In words, W-MON requires that, if player i may change the social outcome from x to y by changing his type declaration from v_i to v_i', then it must be the case that i's value *difference* for y is at least as large as i's value difference for x. The use of value differences (instead of the more intuitive use of absolute values) in the definition comes from the quasi-linearity of utilities, and is best explained by looking at the proof of the following claim:

Lemma 1 *Every dominant-strategy implementable social choice function f satisfies W-MON.*

Proof: We first claim that the payment function of player i can be denoted by $p_i : V_{-i} \times A \rightarrow \Re \cup \infty$, i.e. that p_i does not depend on v_i. Otherwise there are $v_{-i} \in V_{-i},\ x \in A$, and $v_i, v'_i \in V_i$ such that $f(v_i, v_{-i}) = f(v'_i, v_{-i}) = x$, and $p_i(v_i, v_{-i}) < p_i(v'_i, v_{-i})$. But then, when all other players declare v_{-i} and the true type of i is v'_i, he will increase his utility by declaring v_i, a contradiction.

Now fix some $v_{-i} \in V_{-i}$ and $v_i,\ v'_i \in V_i$ as in the definition of WMON. It must be the case that $v_i(x) - p_i(x, v_{-i}) \geq v_i(y) - p_i(y, v_{-i})$, otherwise when i has type v_i he can increase his utility by misreporting his type to be v'_i (and by this causing y to be chosen). Similarly, $v'_i(y) - p_i(y, v_{-i}) \geq v'_i(x) - p_i(x, v_{-i})$. Combining these inequalities we get $v'_i(y) - v_i(y) \geq v'_i(x) - v_i(x)$, as W-MON requires. ∎

Thus, W-MON is necessary for implementability on *every* domain. Quite interestingly, it is also sufficient on many domains. Bikhchandani et. al. [2] show the sufficiency of W-MON for a certain family of auction domains (that includes, as a special case, the unrestricted domain), and Saks and Yu [13] extend the result to all convex domains. In the proofs here, we do not use W-MON directly. Instead we use two closely related conditions, that are easier to work with in our unrestricted domain, and that we next describe.

2.1 PAD

Roberts, in his original proof, formalizes and uses the following "Positive Association of Differences" (PAD) monotonicity condition:

Definition 2 (Roberts [10]) *A social choice function f satisfies* **PAD** *if the following holds for all $v, v' \in V$: If $f(v) = x$, and $v'_i(x) - v_i(x) > v'_i(y) - v_i(y)$ for all $y \in A \setminus x$ and all $i = 1, \ldots, n$, then it must be the case that $f(v') = x$, as well.*

PAD continues to focus on value differences, as W-MON does, but makes all players symmetric, in the shift from v to v'. PAD easily follows from W-MON, as we next show, and is hence necessary for dominant-strategy implementability:

Lemma 2 *Every implementable social choice function f satisfies PAD.*

Proof: By Lemma 1, f satisfies W-MON. Fix some $v, v' \in V$ as in the definition of PAD, i.e. $f(v) = x$ and $v'_i(x) - v_i(x) > v'_i(y) - v_i(y)$ for all $y \in A \setminus x$ and all $i = 1, \ldots, n$. We need to show that $f(v') = x$. Denote $v^i = (v'_1, \ldots, v'_i, v_{i+1}, \ldots, v_n)$ (i.e. all players up to i declare according to v', and the rest declare according to v). Thus $v^0 = v$ and $v^n = v'$, and $f(v^0) = x$. We show by induction that $f(v^n) = x$. Assume by contradiction that $f(v^{i-1}) = x$ but $f(v^i) = y$. Since all players except player i have the same type in v^{i-1} and in v^i, we get by W-MON that $v'_i(y) - v_i(y) \geq v'_i(x) - v_i(x)$, a contradiction to PAD's assumption on v, v'. So $f(v^i) = x$. By induction, $f(v^n) = x$. ∎

In addition to the direct use of PAD, we sometimes use it also implicitly, through the following claim:

Notation: Given vectors $\alpha, \beta \in \Re^n$, we use $\alpha > \beta$ to denote that $\forall i,\ \alpha_i > \beta_i$, i.e., a strict inequality in every coordinate. Additionally, we shall denote $\vec{0}$ as the n-dimensional vector of zeros.

Claim 1 *Assume f satisfies PAD, and fix any $v, v' \in V$. If $f(v') = x$ and $v'(y) - v(y) > v'(x) - v(x)$ for some $y \in A$, then $f(v) \neq y$.*

Proof: Suppose by contradiction that $f(v) = y$. Fix $\Delta \in \Re^n$ such that $\Delta = v'(y) - v(y) + v(x) - v'(x)$. Clearly, $\Delta_i > 0$ for all i. Additionally, $v_i(x) - v'_i(x) - \frac{\Delta_i}{2} = v_i(y) - v'_i(y) + \frac{\Delta_i}{2} > v_i(y) - v'_i(y)$. Define a new type $v'' \in V$ as follows:

$$\forall i,\ z \in A\ :\ v''_i(z) = \begin{cases} \min\{v_i(z)\ ,\ v'_i(z) + v_i(x) - v'_i(x)\} - \Delta_i & z \neq x, y \\ v_i(x) - \frac{\Delta_i}{2} & z = x \\ v_i(y) & z = y. \end{cases}$$

We will show that, from PAD, the transition $v \to v''$ implies $f(v'') = y$, and the transition $v' \to v''$ implies $f(v'') = x$, a contradiction.

Since $v''_i(y) - v_i(y) = 0 > v''_i(z) - v_i(z)$ for all $z \neq y$ and all i, it follows from PAD that $f(v'') = y$. On the other hand, for $z \neq x, y$, $v''_i(z) \leq v'_i(z) + v_i(x) - v'_i(x) - \Delta_i$ and thus $v''_i(x) - v'_i(x) = v_i(x) - v'_i(x) - \frac{\Delta_i}{2} > v''_i(z) - v'_i(z)$. For y, $v''_i(x) - v'_i(x) = v_i(x) - v'_i(x) - \frac{\Delta_i}{2} > v_i(y) - v'_i(y) = v''_i(y) - v'_i(y)$, and thus it also follows from PAD that $f(v'') = x$. ■

2.2 S-MON

Our second proof utilizes a different extension of W-MON, termed Strong Monotonicity (S-MON). This extension requires the inequality in the monotonicity condition to be strict:

Definition 3 (Lavi, Mu'alem, and Nisan [7]) *A social choice function $f : V \to A$ satisfies* **S-MON** *if $f(v) = x$, $f(v'_i,\ v_{-i}) = y$ and $x \neq y$, implies that:*

$$v'_i(y) - v_i(y) > v'_i(x) - v_i(x)$$

for all $v_{-i},\ v_i,\ v'_i \in V_i$.

Thus, the only difference between W-MON and S-MON is the strictness of the inequality. While this may seem as a minor difference, a "tie-breaking" consistency, it is actually much more subtle, and [7] give examples for implementable functions that satisfy W-MON, but violate S-MON in a fundamental way that cannot be viewed as a "tie-breaking" difference. However, all these examples use *restricted* domains, and this is not an accident. For the unrestricted domain, [7] indeed show that the immediate tie-breaking intuition can be formalized and made exact, in the following way.

Theorem 2 (Lavi, Mu'alem, and Nisan [7]) *Suppose V is an open set [1]. Then for every social choice function $f : V \to A$ there exists a social choice function $\tilde{f} : V \to A$ such that:*

1. *If f satisfies W-MON then $\tilde{f}$ satisfies S-MON.*
2. *If $\tilde{f}$ is affine maximizer then f is affine maximizer.*

[1] We use the term "open" in its usual topological meaning: V is open if for any $v \in V$ there exists $\epsilon_v > 0$ such that for any $v' \in \mathcal{R}^{|A| \times n}$, if $|v'_i(x) - v_i(x)| < \epsilon_v$ for all i, x then $v' \in V$ as well.

Since the unrestricted domain is open, then, by this theorem, proving that S-MON implies affine maximization exactly implies that W-MON implies affine maximization: Using the first step of the theorem we "generate" from f that satisfies W-MON an $\tilde{f}$ that satisfies S-MON. We then show that this $\tilde{f}$ is an affine maximizer using the main theorem. Finally, by the second step of Theorem 2 we conclude that the original f is also an affine maximizer.

The proof of theorem 2 exactly utilizes the tie-breaking intuition. Let us use the notation $v+\epsilon 1_x$ to denote the valuation that is almost identical to v, except that, for every i, $v_i(x)$ is increased by ϵ. Then, for a valuation v, the following set of alternatives $T(v)$ captures all the alternatives that are "tied" with $f(v)$:

$$T(v) \;=\; \{\; x \in A \mid \exists \epsilon^* > 0 \text{ s.t. } \forall \epsilon \in (0, \epsilon^*) : f(v + \epsilon 1_x) = x \;\}$$

The function $\tilde{f}$ that fixes some predetermined order over all the alternatives, and always chooses the highest alternative in $T(v)$ to be $\tilde{f}$, can be shown to satisfy the two requirements of Theorem 2. The first property follows by showing that, for any $v \in V$, i, and $v'_i \in V_i$: if $x \in T(v)$, $y \in T(v'_i, v_{-i})$, and $v'_i(x) - v_i(x) \geq v'_i(y) - v_i(y)$, then $x \in T(v'_i, v_{-i})$ (and this in turn follows by applying W-MON). The second property follows in a straight-forward way from the definition of affine maximization.

Our theorem that S-MON implies affine maximization (theorem 3) also requires the social choice function to be implementable, and to satisfy "decisiveness": for every v_{-i} and every $x \in A$ there exists v_i such that $f(v_i, v_{-i}) = x$. Therefore, in order to use $\tilde{f}$ instead of f in the theorem, we need to show that the transition from f to $\tilde{f}$ carries these two properties. Indeed, $\tilde{f}$ is implementable since it satisfies S-MON, and therefore also W-MON, which is a sufficient condition for implementability [2, 13]. It is not hard to verify that $\tilde{f}$ is also decisive, by the following argument: Fix some $\epsilon_0 > 0$. We are given any v_{-i} and x, and need to find v_i such that $\tilde{f}(v_i, v_{-i}) = x$. Let $v'_{-i} = v_{-i} - \epsilon_0 1_x$. Since f is decisive, there exists v'_i such that $f(v'_i, v'_{-i}) = x$. Let $v_i = v'_i + \epsilon_0 1_x$. By PAD, it follows that $T(v_i, v_{-i}) = \{x\}$, since for every $\epsilon < \epsilon_0$ and every $y \neq x$, $f(v_i + \epsilon 1_y, v_{-i} + \epsilon 1_y) = x$. Therefore $\tilde{f} = x$, as required.

Thus, although replacing W-MON by S-MON requires some technical preparations, our proof shows that it can significantly simplify the "rest" of the analysis (if one is willing to additionally assume decisiveness). Since the W-MON – S-MON "equivalence" does not hold for most *restricted* domains, one can view the second proof as an indication that similar impossibilities for restricted domains crucially depend on the differentiation between S-MON and W-MON, an indication that also appears in [7].

3 First Proof

The starting point of both proofs is the simple, but important observation that, to show that a function is an affine maximizer, one should actually study the structure of valuation *differences*. This comes from the fact that affine maximization translates to the system of inequalities

$$\sum_{i=1..n} k_i \cdot (v_i(x) - v_i(y)) \geq C_y - C_x,$$

where v is such that $f(v) = x \neq y$. Conveniently, all the above monotonicity properties are indeed expressed by valuation differences. The second proof fixes a player, i, and the declaration v_{-i} of the other players, and studies the resulting structure of valuation differences for player i, to show that it yields the above system of inequalities. In contrast, the first proof *aggregates* the value differences

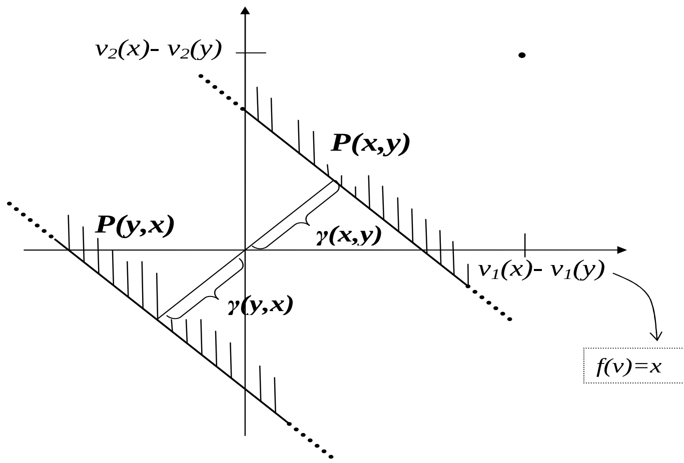

Figure 1: The structure of $P(x, y)$ and $P(y, x)$ (for two players), as the proof reveals. The set $P(x, y)$ is defined to contain all points $(v_1(x) - v_1(y), v_2(x) - v_2(y))$ such that $f(v_1, v_2) = x$, e.g. the black dot in the north east corner of the Figure.

of all players, and studies the topological structures induced by *vectors* of value differences. The main topological set that is being explored is given by:

$$P(x, y) = \{\alpha \in \Re^n \mid \exists v \in V \text{ such that } v(x) - v(y) = \alpha \text{ and } f(v) = x\}.$$

In words, if $f(v) = x$ then $v(x) - v(y) \in P(x, y)$. Figure 1 describes the structure of these sets, as the proof reveals. It shows that $P(x, y)$ is a half-space shifted by $\gamma(x, y)$ from the center of axes, where $\gamma(x, y)$ is the smallest shift needed for the origin to be on the boundary of $P(x, y)$. The slope of this half-space, relative to the center of axes, exactly gives us the constants k_i's in the affine maximization term. These k_i's do not depend on the tuple (x, y), as the proof shows. The shift factors give us the additive constants $\{C_x\}_{x \in A}$. Roberts' original proof uses the same $P(x, y)$ sets (and we have kept the original notation to make this point explicit). However, in order to reveal the topological structure of these $P(x, y)$ sets, the original proof relies on certain properties of an underlying complete order over the set of alternatives, as well as on other auxiliary constructions (e.g. the so-called $Q(x, y)$ sets). The analysis we give here is more concise, and shows with less effort that these $P(x, y)$ sets are half-spaces.

Briefly and informally, the two main observations about these sets are: (1) $\alpha \in P(x, y)$ if and only if $-\alpha \notin P(y, x)$, and (2) $P(x, y) + P(y, z) = P(x, z)$ (to be accurate, these properties hold only for the *interiors* of the $P(x, y)$ sets). To get some geometric intuition to the implications of these two statements, assume for a minute that $\vec{0} \in P(x, y)$ for all $x, y \in A$ (this assumption need not necessarily hold, and in the proof, like in Roberts' proof, the sets are "shifted", as explained above, to account for that). Then the following short chain of arguments leads us to the desired conclusion:

1. Property (2) implies that all the $P(x, y)$ sets are equal (since $\alpha \in P(x, y)$ implies that $\alpha + \vec{0} \in P(x, z)$). Let us denote this set by C.

2. Property (1) then implies that C is convex: assume by contradiction that $\alpha, \beta \in C$, but

$\frac{1}{2}(\alpha+\beta) \notin C$. Then property (1) implies that $-\frac{1}{2}(\alpha+\beta) \in C$. Property (2) implies that $\alpha+\beta \in C$, and therefore $\alpha+\beta-\frac{1}{2}(\alpha+\beta) \in C$ as well, a contradiction.

3. Let $-C = \{ \alpha \mid -\alpha \in C \}$. Property (1) implies that $C \cup -C = \Re^n$: If $\alpha \notin C$ then $-\alpha \in C$ and therefore $\alpha \in -C$.

4. Since C and $-C$ are convex, with disjoint interiors, and their union is the entire space, each must be a half-space.

Thus, these two basic observations about the $P(x,y)$ sets significantly simplify the analysis, and we give a very short and straight-forward proof for their correctness. Roberts' original proof essentially entails this structure, but it is practically lost in the many additional constructions that are being used there. Here, the simplicity of these arguments is emphasized.

3.1 The proof

As described above, the proof studies the following sets, which are defined for tuples $(x,y) \in A \times A$, $x \neq y$:

$$P(x,y) = \{\alpha \in \Re^n \mid \exists v \in V \text{ such that } v(x)-v(y)=\alpha \text{ and } f(v)=x\}. \tag{1}$$

Figure 1 describes the structure of these sets, as the following proof will reveal. We start with two immediate properties:

1. For every x and y, the set $P(x,y)$ is not empty, since by assumption f is onto A.

2. If $\alpha \in P(x,y)$ then for any positive $\delta \in \Re^n$ (i.e. $\delta > \vec{0}$), $\alpha+\delta \in P(x,y)$. To see this, note that by definition there exists v such that $f(v)=x$ and $v(x)-v(y)=\alpha$. Now change v to v' by increasing $v(x)$ by δ (the rest stays the same). Thus by PAD $f(v')=x$, and $v'(x)-v'(y)=\alpha+\delta$, as required.

Claim 2 *For every $\alpha, \epsilon \in \Re^n$, $\epsilon > \vec{0}$:*

1. $\alpha-\epsilon \in P(x,y) \quad \Rightarrow \quad -\alpha \notin P(y,x)$.

2. $\alpha \notin P(x,y) \quad \Rightarrow \quad -\alpha \in P(y,x)$.

Proof: For the first part, suppose by contradiction that $-\alpha \in P(y,x)$. Therefore there exists $v \in V$ with $v(y)-v(x)=-\alpha$ and $f(v)=y$. As $\alpha-\epsilon \in P(x,y)$, there also exists $v' \in V$ with $v'(x)-v'(y)=\alpha-\epsilon$ and $f(v')=x$. But since $v(x)-v(y)=\alpha > v'(x)-v'(y)$, this contradicts claim 1. Thus $-\alpha \notin P(y,x)$, as needed.

For the second part, for any $z \neq x,y$ take some $\beta_z \in P(x,z)$ and fix some $\epsilon > \vec{0}$. Choose any v such that $v(x)-v(y)=\alpha$ and $v(x)-v(z)=\beta_z+\epsilon$ for all $z \neq x,y$. Applying claim 1 for every $z \neq x,y$, we get that $f(v) \in \{x,\ y\}$. Since $v(x)-v(y)=\alpha \notin P(x,y)$ it follows that $f(v)=y$. Thus $-\alpha = v(y)-v(x) \in P(y,x)$, as needed. ∎

As a checkpoint, we specify the parts of the view outlined in Figure 1 that are now proved by the above arguments: The second "immediate property" from above shows that the boundary of $P(x,y)$ is monotonically non-increasing. Claim 2 exactly shows that the interiors of $P(y,x)$ and the "mirror set" of $P(x,y)$ are disjoint, and that their union forms the entire space (where the mirror set of $P(x, y)$ contains $-\alpha$ if and only if α is in $P(x, y)$). Indeed, setting $\gamma(x, y) = -\gamma(y, x)$ in Figure 1 would imply exactly that. We do not know, yet, that the boundaries of these sets are hyperplanes, and the rest of the proof aims to show this issue.

Claim 3 *For every α, β, $\epsilon^{(\alpha)}$, $\epsilon^{(\beta)}$, $\in \Re^n$, $\epsilon^{(\alpha)}$, $\epsilon^{(\beta)} > \vec{0}$:*

$$\alpha - \epsilon^{(\alpha)} \in P(x, y) \text{ and } \beta - \epsilon^{(\beta)} \in P(y, z) \quad \Rightarrow \quad \alpha + \beta - (\epsilon^{(\alpha)} + \epsilon^{(\beta)})/2 \in P(x, z).$$

Proof: For any $w \neq x, y, z$ fix some $\delta^{(w)} \in P(x, w)$, and some $\epsilon \in \Re^n, \epsilon > \vec{0}$. Choose any v such that $v(x) - v(y) = \alpha - \epsilon^{(\alpha)}/2$, $v(y) - v(z) = \beta - \epsilon^{(\beta)}/2$, and $v(x) - v(w) = \delta^{(w)} + \epsilon$ for all $w \neq x, y, z$. By claim 1, $f(v) = x$. Thus $\alpha + \beta - (\epsilon^{(\alpha)} + \epsilon^{(\beta)})/2 = v(x) - v(z) \in P(x, z)$. ∎

If we had $\vec{0} \in P(x, y)$ for all $x, y \in A$ then we could have easily concluded, by using the last claim, that all the $P(x, y)$ sets are equal (to be accurate, only their interiors); this essentially follows by claiming that $P(x, y) \subseteq P(x, z)$, since for any α in the interior of $P(x, y)$ we have that $\alpha + \vec{0} \in P(y, z)$ by the last claim. However, since it is not necessarily true that $\vec{0} \in P(x, y)$, we need to "shift" the sets to contain this point. We do this using the following definition:

$$\gamma(x, y) = \inf\{p \in \Re \mid p \cdot \vec{1} \in P(x, y) \}.$$

It is easy to show that $\gamma(x, y)$ is a real number: First, we argue that the set is not empty. Take any v such that $f(v) = x$, let $p = max_i\{v_i(x) - v_i(y)\}$ and increase all coordinates $v_i(x)$ until $v_i(x) - v_i(y) = p$ for all i. By PAD the outcome remains x. This shows that $\gamma(x, y) < \infty$. Second, the set is also bounded from below, otherwise by claim 2 $P(y, x)$ would have been empty. Thus the set has a real infimum.

Claim 4 *For all $x, y, z \in A$, the following holds:*

1. $\gamma(x, y) = -\gamma(y, x)$.
2. $\gamma(x, z) = \gamma(x, y) + \gamma(y, z)$.

Proof: For the first part, assume that $\gamma(x, y) = p^*$. Thus for any $\epsilon > 0$, $(p^* + (\epsilon/2)) \cdot \vec{1} \in P(x, y)$ and so by claim 2, $(-p^* - \epsilon) \cdot \vec{1} \notin P(y, x)$. On the other hand by the definition of $\gamma(x, y)$, $(p^* - \epsilon) \cdot \vec{1} \notin P(x, y)$ and thus by claim 2, $(-p^* + \epsilon) \cdot \vec{1} \in P(y, x)$. Thus $-p^* = \inf\{p \mid p \cdot \vec{1} \in P(y, x) \}$, as needed.

For the second part, fix some $\epsilon > 0$. Since $(\gamma(x, y) + (\epsilon/2)) \cdot \vec{1} \in P(x, y)$, $(\gamma(y, z) + (\epsilon/2)) \cdot \vec{1} \in P(y, z)$ it follows from claim 3 that $(\gamma(x, y) + \gamma(y, z) + \epsilon) \cdot \vec{1} \in P(x, z)$ and thus $\gamma(x, z) \leq \gamma(x, y) + \gamma(y, z)$. By exchanging variable names, we also get $\gamma(z, x) \leq \gamma(z, y) + \gamma(y, x)$. Replacing $\gamma(y, x)$ with $-\gamma(x, y)$ (and doing this also for $\gamma(z, x)$, $\gamma(z, y)$) we get that $\gamma(x, z) \geq \gamma(x, y) + \gamma(y, z)$ as well, and the claim follows. ∎

We can now "shift" the sets $P(x, y)$ by defining:

$$C(x, y) = P(x, y) - \gamma(x, y) \cdot \vec{1}.$$

More precisely, $C(x, y) = \{\alpha - \gamma(x, y) \cdot \vec{1} \mid \alpha \in P(x, y) \}$. Let $\overset{\circ}{C}(x, y)$ denote the interior of $C(x, y)$, i.e. $\overset{\circ}{C} = \{\alpha \in C \mid \alpha - \epsilon \in C \text{ for some } \epsilon > \vec{0}\}$.

Claim 5 *$\overset{\circ}{C}(x,y) = \overset{\circ}{C}(w,z)$, for every $x,y,w,z \in A$, $x \neq y$ and $w \neq z$.*

Proof: By claim 3, $\overset{\circ}{P}(x,y) \subseteq \overset{\circ}{P}(x,z) - \beta$, for any $\beta \in \overset{\circ}{P}(y,z)$, specifically for $\beta' = (\gamma(y,z)+\epsilon)\cdot\vec{1}$ for any $\epsilon > 0$. Similarly (by exchanging variable names in the claim), $\overset{\circ}{P}(x,z) \subseteq \overset{\circ}{P}(w,z) - \alpha$, for any $\alpha \in \overset{\circ}{P}(w,x)$, specifically for $\alpha' = (\gamma(w,x)+\epsilon)\cdot\vec{1}$. Thus $\overset{\circ}{P}(x,y) \subseteq \overset{\circ}{P}(w,z) - (\gamma(y,z)+\gamma(w,x))\cdot\vec{1}$. By claim 4, $\gamma(y,z)+\gamma(w,x) = \gamma(y,z)+\gamma(w,y)+\gamma(y,x) = \gamma(w,z)-\gamma(x,y)$. And so, $\overset{\circ}{P}(x,y)-\gamma(x,y)\cdot\vec{1} \subseteq \overset{\circ}{P}(w,z) - \gamma(w,z)\cdot\vec{1}$, as claimed. ∎

Remark: When x,y,w,z are not distinct the claim still holds. For example, $\overset{\circ}{C}(x,y) = \overset{\circ}{C}(y,x)$. For this to hold we need a third (distinct) alternative w, and then $\overset{\circ}{C}(x,y) = \overset{\circ}{C}(w,y) = \overset{\circ}{C}(w,x) = \overset{\circ}{C}(y,x)$ in the same manner as done in the proof.

As a result of the last claim, we denote $C = \overset{\circ}{C}(x,y) = \overset{\circ}{C}(w,z)$. To finish the proof, we first conclude that C is convex:

Claim 6 *C is convex.*

Proof: We will show that for any $\alpha,\beta \in C \subseteq \Re^n$, it must be the case that $(\alpha+\beta)/2 \in C$. Since C is open, it is well known that all together this implies that C is convex[2]. We first show that $\alpha+\beta \in C$. Fix some distinct $x,y,z \in A$. Since $\gamma(x,y)\cdot\vec{1}+\alpha \in \overset{\circ}{P}(x,y)$ and $\gamma(y,z)\cdot\vec{1}+\beta \in \overset{\circ}{P}(y,z)$ it follows from claims 3 and 4 that $\gamma(x,z)\cdot\vec{1}+\alpha+\beta \in \overset{\circ}{P}(x,z)$, and thus $\alpha+\beta \in C$.

We now show that for any $\alpha \in C$ we have $\alpha/2 \in C$ as well. Assume by contradiction that $\alpha/2 \notin C$. Thus $\alpha/2 + \gamma(x,y)\cdot\vec{1} \notin P(x,y)$ and by claim 2, $-\alpha/2 - \gamma(x,y)\cdot\vec{1} \in P(y,x)$. Since $-\gamma(x,y) = \gamma(y,x)$ it follows that $-\alpha/2 \in C$. But then $\alpha/2 = \alpha + (-\alpha/2) \in C$ by using the above claim, and we reach a contradiction. ∎

We can now easily conclude the proof of the theorem. Notice that $\vec{0} \notin \overset{\circ}{C}$ (if $\vec{0} \in C(x,y)$ then it must be on its boundary since $(\gamma(x,y)-\epsilon)\cdot\vec{1} \notin P(x,y)$ for any $\epsilon > 0$). By the Separating Hyperplane Lemma, there exists $k \in \Re^n$ such that for any $\alpha \in \bar{C}$, $k\cdot\alpha \geq 0$ (where $\bar{C}$ is the closure of C). Now, fix some $x_0 \in A$ and determine the constants C_x for any $x \in A$ as $C_x = \sum_{i=1}^n k_i \cdot \gamma(x_0,x)$ (define $\gamma(x_0,x_0) \equiv 0$). To see that the theorem now follows with these constants, suppose $f(v) = x$ for some $v \in V$, and take any $y \neq x$. Clearly, $v(x) - v(y) \in P(x,y)$. Let $\alpha = v(x) - v(y) - \gamma(x,y)\cdot\vec{1}$, and so $\alpha \in \bar{C}$. Thus $k\cdot\alpha \geq 0$. Replacing $-\gamma(x,y) = \gamma(x_0,x) - \gamma(x_0,y)$ and rearranging the terms we get: $k\cdot v(x) + C_x \geq k\cdot v(y) + C_y$, and the theorem follows. ∎

4 Second Proof

As described previously, the second proof starts from a different monotonicity condition, and uses a completely different analysis method. We need to assume one additional condition, defined by Meyer-ter-Vehn and Moldovanu [8]:

[2] For any $\alpha,\beta \in C$ and $0 \leq \lambda \leq 1$, the argument essentially builds a series of points that approach $\lambda\alpha + (1-\lambda)\beta$, with the property that any point in the series has a ball of some fixed radius around it that fully belongs to C.

Definition 4 (Player Decisiveness) *Given a social choice function $f : V \to A$, we say that player i is decisive if for every $v_{-i} \in V_{-i}$ and $x \in A$, there exists $v_i \in V_i$ such that $f(v_i, v_{-i}) = x$.*

In other words, given the types of the other players, player i can enforce the choice of any alternative (e.g. by assigning a "high enough" value to it). It is interesting to note that when only two possible alternatives exist, the majority rule (i.e. player 1 "votes" for the alternative x with highest value) is implementable, but is not decisive. For three or more alternatives, we know by the previous proof that every implementable social choice function must admit at least one decisive player, but we do not know of an "easy" explicit way of proving this. Our proof explicitly requires the existence of *one* decisive player.

Theorem 3 *Suppose V is unrestricted and $|A| \geq 3$. Then every implementable social choice function f with at least one decisive player has non-negative constants $k_1, \ldots, k_n$, not all of them equal to zero, and constants $\{C_x\}_{x \in A}$ such that, for all $v \in V$,*

$$f(v) \in \mathit{argmax}_{x \in A} \ \{ \ \Sigma_{i=1}^{n} k_i v_i(x) + C_x \ \}.$$

From now on we assume w.l.o.g. that player 1 is decisive. Using Theorem 2, we can additionally assume w.l.o.g that f satisfies S-MON. Throughout the proof we will use the following notation:

Notation: The valuation $v' = v + \epsilon \cdot 1_{i,x}$, or equivalently $v' = (v_i + \epsilon \cdot 1_x, \ v_{-i})$, is almost identical to v, except that $v_i(x)$ is increased by ϵ. Finally, e_j denotes the j-th elementary vector.

The proof in this section analyzes the implications of the following basic notion. This notion stems from the work of Rochet [11] (and, independently, [12]), and is tightly related to W-MON in a way that will be discussed below.

Definition 5 *For every two distinct $x, y \in A$, and any $v_{-i} \in V_{-i}$, define:*

$$\delta^i_{xy}(v_{-i}) = \inf \ \{v'_i(x) - v'_i(y) \mid v'_i \in V_i \ \text{such that} \ f(v'_i, v_{-i}) = x\}.$$

By definition, if $f(v) = x$, then $v_i(x) - v_i(y) \geq \delta^i_{xy}(v_{-i})$ for every $y \in A$. In words, fixing v_{-i}, $\delta^i_{xy}(v_{-i})$ is the minimal value difference between x and y whenever f chooses x. For instance, in a second price auction, $\delta^i_{\text{win}_i, \text{lose}_i}(v_{-i})$ is equal to the minimal value player i needs to declare in order to win the item. The next claims explore the structural properties of this definition, for the case of an unrestricted domain, and show that $\delta^1_{xy}(v_{-1})$ is an affine function of the vector of value differences $v_{-1}(x) - v_{-1}(y)$. From this, the affine maximization property will follow.

Claim 7 *For every $v_{-1} \in V_{-1}, \ x, y \in A, \ \delta^1_{xy}(v_{-1})$ is a real number, and $\delta^1_{xy}(v_{-1}) + \delta^1_{yx}(v_{-1}) \geq 0$.*

Proof: By decisiveness, there exists $v_1 \in V_1$ such that $f(v_1, v_{-1}) = x$. Thus $\delta^1_{xy}(v_{-1}) \leq v_1(x) - v_1(y) < \infty$. Additionally, decisiveness also implies that there exists v^*_1 such that $f(v^*_1, v_{-1}) = y$. For any $v'_1 \in V_1$ with $f(v'_1, v_{-1}) = x$ we have, by W-MON, that $v'_1(x) - v'_1(y) \geq v^*_1(x) - v^*_1(y)$. Hence $\delta^1_{xy}(v_{-i}) \geq v^*_1(x) - v^*_1(y) > -\infty$.

For the second part, fix any $\epsilon > 0$, and take $v^*_1 \in V_1$ such that $f(v^*_1, v_{-1}) = y$ and $v^*_1(y) - v^*_1(x) \leq \delta^1_{yx}(v_{-1}) + \epsilon$, and $v'_1 \in V_1$ with $f(v'_1, v_{-1}) = x$ and $v'_1(x) - v'_1(y) \leq \delta^1_{xy}(v_{-1}) + \epsilon$. By W-MON we have $v'_1(x) - v'_1(y) \geq v^*_1(x) - v^*_1(y)$. Thus $\delta^1_{xy}(v_{-1}) + \epsilon \geq v'_1(x) - v'_1(y) \geq v^*_1(x) - v^*_1(y) \geq -\delta^1_{yx}(v_{-1}) - \epsilon$.

Therefore $\delta^1_{xy}(v_{-1}) + \delta^1_{yx}(v_{-1}) + 2 \cdot \epsilon \geq 0$ for any $\epsilon > 0$, and the claim follows. ∎

The inequality $\delta^i_{xy}(v_{-i})+\delta^i_{yx}(v_{-i}) \geq 0$ was interpreted by Gui, Muller, and Vohra [3] as a 2-cycle inequality, in the context of a specific "allocation graph". For convex domains, the non-negativity of all 2-cycles implies implementability [13]. For general domains, this is not necessarily the case, and Rochet [11] replaced the 2-cycle requirement with an "all possible cycles" requirement, showing that this is equivalent to implementability on *all* domains. Our proof shows that, in an unrestricted domain, these δ^1's have a concrete and well-defined structure, that essentially translates to being affine functions.

Claim 8 *For every $v_{-1} \in V_{-1},\ x, y \in A,\ \ \delta^1_{xy}(v_{-1}) + \delta^1_{yx}(v_{-1}) = 0$.*

Proof: By the previous claim it is enough to show that $\delta^1_{xy}(v_{-1}) + \delta^1_{yx}(v_{-1}) \leq 0$. For every $\epsilon \geq 0$ and v_1 such that $f(v_1,\ v_{-1}) = x$ and $v_1(x) - v_1(y) = \epsilon + \delta^1_{xy}(v_{-1})$, consider $v'_1 = v_1 + 3\epsilon \cdot 1_y + \epsilon \cdot 1_x$. Then $f(v'_1,\ v_{-1}) \in \{x, y\}$ by W-MON. However, $f(v'_1,\ v_{-1})$ cannot be x, since $v'_1(x) - v'_1(y) = (v_1(x)+\epsilon) - (v_1(y)+3\epsilon) < \delta^1_{xy}(v_{-1})$. We get that $f(v'_1,\ v_{-1}) = y$. Thus $\delta^1_{yx}(v_{-1}) \leq v'_1(y) - v'_1(x) = v_1(y) - v_1(x) + 2\epsilon = -\delta^1_{xy}(v_{-1}) + \epsilon$, and so $\delta^1_{xy}(v_{-1}) + \delta^1_{yx}(v_{-1}) \leq \epsilon$ for every $\epsilon \geq 0$. ∎

Claim 9 *For every $v_{-1} \in V_{-1},\ x, y$ and $z \in A,\ \ \delta^1_{xy}(v_{-1}) + \delta^1_{yz}(v_{-1}) + \delta^1_{zx}(v_{-1}) = 0$.*

Proof: Fix v_{-1}. For every v_1, v'_1, v''_1 such that $f(v_1,\ v_{-1}) = x,\ f(v'_1,\ v_{-1}) = y,\ f(v''_1,\ v_{-1}) = z$, truthfulness implies that $v_1(x) - p_1(x, v_{-1}) \geq v_1(y) - p_1(y, v_{-1}),\ v'_1(y) - p_1(y, v_{-1}) \geq v'_1(z) - p_1(z, v_{-1})$, and $v''_1(z) - p_1(z, v_{-1}) \geq v''_1(x) - p_1(x, v_{-1})$. Thus $v_1(x) - v_1(y) + v'_1(y) - v'_1(z) + v'_1(z) - v'_1(x) \geq 0$. In particular, $\delta^1_{xy}(v_{-1}) + \delta^1_{yz}(v_{-1}) + \delta^1_{zx}(v_{-1}) \geq 0$ [3].

Now, suppose there exist v_{-1}, x, y and z s.t. $\delta^1_{xy}(v_{-1}) + \delta^1_{yz}(v_{-1}) + \delta^1_{zx}(v_{-1}) > 0$. By claim 8 $[\delta^1_{xy}(v_{-1})+\delta^1_{yx}(v_{-1})]+[\delta^1_{yz}(v_{-1})+\delta^1_{zy}(v_{-1})]+[\delta^1_{zx}(v_{-i})+\delta^1_{xz}(v_{-1})] = 0$. And so, $\delta^1_{xz}(v_{-1})+\delta^1_{zy}(v_{-1})+\delta^1_{yx}(v_{-1}) < 0$, a contradiction. ∎

The next two claims show that $\delta^1_{xy}(v_{-1})$ depends only on $v_{-1}(x)-v_{-1}(y)$, the $(n-1)$-dimensional vector that results in taking the difference between $v_{-1}(x)$ and $v_{-1}(y)$. Recall that $(v - \epsilon \cdot 1_{j,z})$ denotes the valuation v in which player j decreases his value for the alternative z by ϵ.

Claim 10 *For every $L \geq 0,\ j \neq 1,\ v_{-1} \in V_{-1}$, and distinct $x, y, z \in A$: $\delta^1_{xy}(v_{-1}) = \delta^1_{xy}(v_{-1} - L \cdot 1_{j,z})$.*

Proof: Let $v'_{-1} = v_{-1} - L \cdot 1_{j,z}$. If $f(v_1,\ v_{-1}) = x$ then S-MON implies that $f(v_1,\ v'_{-1}) = x$, therefore $\delta^1_{xy}(v_{-1}) \geq \delta^1_{xy}(v'_{-1})$. Assume by contradiction that $\delta^1_{xy}(v_{-1}) > \delta^1_{xy}(v'_{-1})$. First note that similar to the above argument, $\delta^1_{yx}(v_{-1}) \geq \delta^1_{yx}(v'_{-1})$. But then (by claim 8) $\delta^1_{xy}(v_{-1}) + \delta^1_{yx}(v_{-1}) = 0 = \delta^1_{xy}(v'_{-1}) + \delta^1_{yx}(v'_{-1})$. By our assumption the LHS is greater than the RHS, a contradiction. ∎

Claim 11 *Let $x, y \in A$ and let v_{-1}, v'_{-1} be such that $v_{-1}(x) - v_{-1}(y) = v'_{-1}(x) - v'_{-1}(y)$. Then $\delta^1_{xy}(v_{-1}) = \delta^1_{xy}(v'_{-1})$.*

Proof: Fix any $v_{-1}, v'_{-1} \in V_{-1}$ with $v_{-1}(x) - v_{-1}(y) = v'_{-1}(x) - v'_{-1}(y)$. For every $j \neq 1$ and every $v_j \in V_j$, S-MON implies that adding a constant to all coordinates of v_j will not

[3] This is actually true for any number of alternatives [11, 12].

change the choice of f. Therefore we can assume w.l.o.g that $v_j(x) = v'_j(x)$ and $v_j(y) = v'_j(y)$. Now define $v''_j(w) = \min\{v_j(w), v'_j(w)\}$ for any $w \in A$. By the last claim (claim 10) we have $\delta^1_{xy}(v_{-1}) = \delta^1_{xy}(v''_{-1}) = \delta^1_{xy}(v'_{-1})$, and the claim follows. ∎

The above claim shows that $\delta^1_{xy}(v_{-1})$ depends only on $v_{-1}(x) - v_{-1}(y)$. From now on we slightly abuse the notation and identify for the ease of exposition $\delta^1_{xy}(v_{-1})$ with $\delta^1_{xy}(v_{-1}(x) - v_{-1}(y))$. The following is immediate:

Conclusion 1 *For every $\bar{r}, \bar{t} \in R^{n-1}$, $x, y, z \in A$: $\delta^1_{xy}(\bar{r}) + \delta^1_{yx}(-\bar{r}) = 0$, and $\delta^1_{xy}(\bar{r}) + \delta^1_{yz}(\bar{t}\,) + \delta^1_{zx}(-\bar{r} - \bar{t}\,) = 0$. In particular, $\delta^1_{xy}(\bar{0}) + \delta^1_{yx}(\bar{0}) = 0$, and $\delta^1_{xy}(\bar{0}) + \delta^1_{yz}(\bar{0}) + \delta^1_{zx}(\bar{0}) = 0$.*

Claim 12 *For every $\bar{r}, \bar{s}, \bar{t} \in R^{n-1}$, $x, y, z \in A$: $\delta^1_{yx}(\bar{r} + \bar{t}) - \delta^1_{yx}(\bar{r}) = \delta^1_{zx}(\bar{s} + \bar{t}) - \delta^1_{zx}(\bar{s})$.*

Proof: It is enough to show that $\delta^1_{zx}(\bar{s}) - \delta^1_{yx}(\bar{r}) = \delta^1_{zx}(\bar{s} + \bar{t}) - \delta^1_{yx}(\bar{r} + \bar{t})$. By conclusion 1, $\delta^1_{zx}(\bar{s}) - \delta^1_{yx}(\bar{r}) = \delta^1_{zx}(\bar{s}) + \delta^1_{xy}(-\bar{r}) = -\delta^1_{yz}(\bar{r} - \bar{s})$. Similarly, $\delta^1_{zx}(\bar{s} + \bar{t}) - \delta^1_{yx}(\bar{r} + \bar{t}) = -\delta^1_{yz}(\bar{r} - \bar{s})$. ∎

Claim 13 *There exist non-negative real constants $k_2, ..., k_n$ such that for every $\bar{r} \in R^{n-1}$ and $y, z \in A$, $\delta^1_{yz}(\bar{r}) = -\Sigma^n_{j=2}\, k_j r_j\ + \delta^1_{yz}(\bar{0})$.*

For the proof of this claim we need the following technical fact (for completeness, we provide the proof for that in the end of the section). Here, a function $g : \Re^n \to \Re$ is monotone if for every $\alpha, \beta \in \Re^n$ with $\beta_i \geq \alpha_i$ for every i, it is the case that $g(\beta) \geq g(\alpha)$.

Technical Claim ([7]): *Fix some monotone function $g : \Re^n \to \Re$, and suppose there exists $h_i : \Re \to \Re$ such that $g(r + \delta \cdot e_i) - g(r) = h_i(\delta)$ for any $r \in \Re^n$ and $\delta > 0$ (where e_i is the i'th unit vector). Then there exist constants $k_i \in \Re$ and $\gamma \in \Re$ such that $g(r) = \sum_{i=1}^n k_i \cdot r_i\ + \gamma$.*

Proof (of claim): First notice that $\delta^1_{yz}(\cdot)$ is a monotone non-increasing real function: If $f(v_1,\ v_{-1}) = y$ then $f(v_1,\ v_{-1} + \epsilon \cdot 1_{j,y}) = y$ by S-MON. And so, the infimum on $v_{-1} + \epsilon \cdot 1_{j,y}$ is taken on a "larger set", and is thus "smaller". Now, claim 12 and the above technical claim together imply that there exist real constants k^{yz}_j such that $\delta^1_{yz}(\bar{r}) = \Sigma_{j \neq 1}\, k^{yz}_j\ r_j + \delta^1_{yz}(\bar{0})$. Since $\delta^1_{yz}(\cdot)$ is monotone non-increasing, each k^{yz}_j must be non-positive. For convenience, we rewrite the equation as $\delta^1_{yz}(\bar{r}) = -\Sigma_{j \neq 1}\, k^{yz}_j\ r_j + \delta^1_{yz}(\bar{0})$, and assume that the k^{yz}_j constants are non-negative. Let us now verify that $k^{xy}_j = k^{wz}_j$ for any $x, y, z, w \in A$. By the above we get that $k^{xy}_j = \delta^1_{xy}(\bar{0}) - \delta^1_{xy}(e_j)$. By conclusion 1 we get $k^{xy}_j = k^{zx}_j$ as $\delta^1_{xy}(e_j) + \delta^1_{yz}(\bar{0}) + \delta^1_{zx}(-e_j) = 0$. Similarly, $k^{zx}_j = k^{wz}_j$. ∎

We can now easily conclude the proof of the theorem. Fix an arbitrary alternative $w \in A$, and set the constants $C_x = \delta^1_{wx}(\bar{0})$ for every $x \neq w$, and $C_w = 0$. Fix $v \in V$, and suppose that $f(v) = x$. Therefore, for every $y \neq x$, $v_1(x) - v_1(y) \geq \delta^1_{xy}(v_{-1}) = -\sum_{j \neq 1} k_j(v_j(x) - v_j(y)) + \delta^1_{xy}(\bar{0})$. Since $\delta^1_{xy}(\bar{0}) = \delta^1_{xw}(\bar{0}) + \delta^1_{wy}(\bar{0})$ and $\delta^1_{xw}(\bar{0}) = -\delta^1_{wx}(\bar{0})$ we get, rearranging terms, that $v_1(x) + \sum_{j \neq 1} k_j v_j(x) + C_x \geq v_1(y) + \sum_{j \neq 1} k_j v_j(y) + C_y$, as needed. ∎

Proof of Technical Claim:

We split the claim to two:

Claim 14 *Suppose $m : \mathcal{R}_+ \to \mathcal{R}$ is monotonically non-decreasing and there exists $h : \mathcal{R}_+ \to \mathcal{R}_+$ such that $m(x+\delta)-m(x) = h(\delta)$ for any $x, \delta \in \mathcal{R}_+$. Then there exist $\omega \in \mathcal{R}_+$ such that $h(\delta) = \omega \cdot \delta$.*

Proof: Let $\omega = h(1)$ (note that $\omega \geq 0$ since m is non-decreasing). First we claim that for any two integers p, q, $h(p/q) = \omega \cdot (p/q)$. Note that $h(1) = m(1) - m(0) = \sum_{i=0}^{q-1} m((i+1)/q) - m(i/q) = q \cdot h(1/q)$. Thus $h(1/q) = (1/q) \cdot h(1)$. Similarly, $h(p/q) = m(p/q) - m(0) = \sum_{i=0}^{p-1} m((i+1)/q) - m(i/q) = p \cdot h(1/q) = (p/q) \cdot h(1) = (p/q) \cdot \omega$. Now we claim that for any real δ, $h(\delta) = \delta \cdot \omega$. Notice that since m is monotonically non-decreasing then h must be monotonically non-decreasing as well. Suppose by contradiction that $h(\delta) > \delta \cdot \omega$. Choose some rational $r > \delta$ close enough to δ such that $h(\delta) > r \cdot \omega$. Since h is monotone and $r > \delta$ then $h(r) \geq h(\delta)$, but since r is rational, $h(r) = r \cdot \omega < h(\delta)$, a contradiction. A similar argument holds if $h(\delta) < \delta \cdot \omega$. ∎

Claim 15 *Suppose that $X \subseteq \mathcal{R}^n$ has the property that $x \in X$ and $y \geq x$ implies $y \in X$. Let $m : X \to \mathcal{R}$ be monotonically non-decreasing, and suppose there exist $\omega_1, \ldots, \omega_n$ such that $m(x + \delta \cdot e_i) - m(x) = \omega_i \cdot \delta$ for any i, $x \in X$, and $\delta > 0$. Then there exist $\gamma \in \mathcal{R}$ such that $m(x) = \sum_{i=1}^n \omega_i \cdot x_i \ + \gamma$.*

Proof: First we claim that for any $x, y \in X$ such that $y_i \geq x_i$ for all i, it is the case that $m(y) = m(x) + \sum_{i=1}^n \omega_i \cdot (y_i - x_i)$. Notice that $(y_1, x_2, \ldots, x_n) \in X$, and $m(y_1, x_2, \ldots, x_n) = m(x) + h_1(y_1 - x_1)$. Repeating this step n times we get $m(y) = m(x) + \sum_{i=1}^n \omega_i \cdot (y_i - x_i)$.

Now fix some $x^* \in X$. We claim that for any $x \in X$, $m(x) = m(x^*) + \sum_{i=1}^n \omega_i \cdot (x_i - x_i^*)$. To see this, choose some y such that $y_i \geq \max\{x_i, x_i^*\}$ for all i. Thus $m(y) = m(x) + \sum_{i=1}^n \omega_i \cdot (y_i - x_i)$ and also $m(y) = m(x^*) + \sum_{i=1}^n \omega_i \cdot (y_i - x_i^*)$, therefore the claim follows. ∎

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